$*$-AUTONOMOUS CATEGORIES IN QUANTUM THEORY

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1. Introduction

So-called $*$-autonomous, or "Frobenius", category structures occur widely in mathematical quantum theory. This trend was observed in [3], mainly in relation to Hopf algebroids, and continued in [8] with a general account of Frobenius monoids.

Below we list some of the $*$-autonomous partially ordered sets $\mathscr{A} = (\mathscr{A}, p, j, S)$ that appear in the literature, an abstract definition of $*$-autonomous promonoidal structure being made in [3, §7]. Without going into much detail, we also note some features of the convolution $[\mathscr{A}, \mathscr{V}]$ (defined in [1]) of a given such $\mathscr{A}$ with a complete $*$-autonomous monoidal category $\mathscr{V}$. A monoidal functor category of this type is a completion of $\mathscr{A}$, with an appropriate universal property; it is always again $*$-autonomous (as seen in [3] for example).

The basic descriptions of promonoidal (equals premonoidal) structure and the resulting convolution product are given in [1] and [3].

2. Examples

The following examples are mostly based in the extended positive real numbers $\mathbb{R}^{\infty}_{\geq 0}$ with the $*$-autonomous monoidal structure of multiplication and identity 1 (we simply define $\infty \otimes 0 = 0$, and $\infty \otimes r = \infty$ for $r \neq 0$).

Remark. The process of adding ∞ to $\mathbb{R}_{\geq 0}$ is quite general. For example, one can add 0 (initial) and ∞ (terminal) to any partially ordered group G and obtain G^{∞}_{0} which is a $*$-autonomous monoidal category. Furthermore, the group G can be replaced by any rigid closed category, etc.

The category $\mathbb{R}^{\infty}_{\geq 0}$ is isomorphic, under exponentiation, to the $*$-autonomous category

$$\mathbb{R}^{\infty}_{-\infty} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

with the monoidal structure of addition and identity 0. In the following, each poset $\mathscr{A}$ is viewed as a category under

$$\mathscr{A}(a, b) = \begin{cases} 1 & \text{iff } a \leq b \\ 0 & \text{else,} \end{cases}$$

and the base category is $\mathscr{V} = \mathbb{R}^{\infty}_{\geq 0}$ unless otherwise mentioned.

Example 1 (Submodular functions [7])**.** Let E be a set and $\mathscr{A} = \mathcal{P}(E)$ (discrete). For $\mathscr{V} = \mathbb{R}^{\infty}_{\geq 0}$, let

$$p(a, b, c) = \begin{cases} 1 & \text{iff } (a \cup b = c \text{ and } a \cap b = \emptyset) \text{ iff } a + b = c \\ 0 & \text{else,} \end{cases}$$

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and

$$j(a) = \begin{cases} 1 & \text{iff } a = \emptyset \\ 0 & \text{else.} \end{cases}$$

Then $(\mathscr{A}, p, j, S)$ becomes a $*$-autonomous promonoidal $\mathscr{V}$-category if we take

$$Sa = E - a,$$

since $p(a, b, Sc) = 1$ iff $a + b = E - c$, i.e., $a + b + c = E$.

Note. In fact Narayanan uses $\mathscr{V} = (\mathbb{R}^{\infty}_{-\infty}, +, 0)$ instead of $\mathbb{R}^{\infty}_{\geq 0}$, and discusses the *upper convolution* on $[\mathscr{A}, \mathscr{V}]$ given by

$$\begin{aligned} f \mathbin{\overline{*}} g(c) &= \sup_{ab} f(a) + g(b) + p(a, b, c) \\ &= \sup_{a \subset c} f(a) + g(c - a). \end{aligned}$$

(Since $\mathscr{A} = \mathcal{P}(E)$ is discrete, $[\mathscr{A}, \mathscr{V}]$ equals all functions from $\mathcal{P}(E)$ to $\mathbb{R}^{\infty}_{-\infty}$, and also the *lower convolution*

$$\begin{aligned} f \mathbin{\underline{*}} g(c) &= (\sup_{ab} f(a)^* + g(b)^* + p(a, b, c))^* \\ &= -(\sup_{ab} -f(a) - g(b) + p(a, b, c)) \\ &= \inf_{a \subset c} f(a) + g(c - a). \end{aligned}$$

Of course in both cases we use the fact that

$$p(a, b, c) = \begin{cases} 0 & \text{iff } a + b = c \\ -\infty & \text{else} \end{cases}$$

in $\mathbb{R}^{\infty}_{-\infty}$.

Note also that $f \in [\mathscr{A}, \mathscr{V}]$ is an upper convolution monoid iff

$$f(a + b) \geq f(a) + f(b) \text{ and } f(0) \geq 0,$$

while f is a lower convolution monoid iff

$$f(a + b) \leq f(a) + f(b) \text{ and } f(0) \leq 0.$$

Example 2. Let $\mathscr{A} = (\mathscr{A}, \vee, \wedge, 0, 1, S)$ be an orthomodular lattice (see Kalmbach [5] for example). Then the definitions

$$p(a, b, c) = \begin{cases} 1 & \text{iff } a \vee b = c \text{ and } a \perp b \\ 0 & \text{else,} \end{cases}$$

$$\mathscr{A}(a, b) = \begin{cases} 1 & \text{iff } a = b \\ 0 & \text{else,} \end{cases}$$

and

$$j(a) = \begin{cases} 1 & \text{iff } a = 0 \\ 0 & \text{else,} \end{cases}$$

yield a (discrete) $*$-autonomous promonoidal poset $(\mathscr{A}, p, j, S)$ where Sa is the orthocomplement of a in $\mathscr{A}$. The orthomodularity condition guarantees the required cyclic relations:

$$p(a, b, Sc) = p(b, c, Sa) = p(c, a, Sb).$$

Example 3 (Browerian logics (Lawvere))**.** Let $(\mathscr{A}, <, \Longrightarrow, \wedge, 0, 1)$ be a Browerian logic. Define an $\mathbb{R}^{\infty}_{\geq 0}$-category by

$$\mathscr{A}(a,b) = \begin{cases} 1 & \text{iff } a < b \\ 0 & \text{else,} \end{cases}$$

and let

$$p(a,b,c) = \begin{cases} 1 & \text{iff } a \wedge b < c \\ 0 & \text{else,} \end{cases}$$

and

$$j(a) = \begin{cases} 1 & \text{iff } a = 1 \\ 0 & \text{else.} \end{cases}$$

If we put $Sa = (a \Longrightarrow -)$, then

$$p(a,b,Sc) = 1 \text{ iff } a \wedge b \wedge c = 0,$$

hence $(\mathscr{A}, p, j, S)$ is a $*$-autonomous promonoidal category.

Example 4 (Groupoids)**.** Let $\mathcal{G}$ be a groupoid and let $\mathscr{A}$ denote the set of arrows of $\mathcal{G}$. Define

$$p(a,b,c) = \begin{cases} 1 & \text{iff } ab = c \\ 0 & \text{else,} \end{cases}$$

$$j(a) = \begin{cases} 1 & \text{iff } a \text{ is an identity} \\ 0 & \text{else,} \end{cases}$$

and $Sa = a^{-1}$. Then $p(a,b,Sc) = 1$ iff $abc = 1$, so $(\mathscr{A}, p, j, S)$ is $*$-autonomous.

Example 5 ((Non-commutative) probabalistic geometry (of [2]))**.** Let $\mathscr{A}$ be a poset with an associative promultiplication

$$p : \mathscr{A}^{\mathrm{op}} \times \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow [0,1] \subset \mathbb{R}^{\infty}_{\geq 0},$$

where we interpret the value $p(a,b,c)$ as the probability that the point c lies in the line through a and b. Here there is no identity j in general. The convolution of poset maps f and g from $\mathscr{A}$ to $[0,1]$ is then the *join* of f to g:

$$f * g = \sup_{ab} f(a) g(b) p(a,b,-),$$

while f is *convex* iff $f * f \leq f$; i.e., iff f is a convolution semigroup.

In particular, note that if $\mathscr{A}$ is discrete, this (A, p) is $*$-autonomous with respect to $Sa = a$ iff

$$p(a,b,c) = p(b,c,a) = p(c,a,b),$$

and these simultaneously take the value 1 iff the points a, b, and c are "collinear".

Example 6 (Generalized effect and difference algebras (cf. Kalmbach [5], Chapter 21; see also [4]))**.** Suppose the poset $\mathscr{A}$ has the structure of a (non-commutative, say) generalized effect algebra $(\mathscr{A}, \oplus, 0, \leq)$. Let

$$p(a,b,c) = \begin{cases} 1 & \text{iff } a \oplus b \leq c \\ 0 & \text{else,} \end{cases}$$

$$\mathscr{A}(a,b) = \begin{cases} 1 & \text{iff } a \leq b \\ 0 & \text{else,} \end{cases}$$

and

$$j(a) = \begin{cases} 1 & \text{iff } 0 \le a \\ 0 & \text{else,} \end{cases}$$

then $(\mathscr{A}, p, j)$ is an (associative and unital) promonoidal category with some extra properties (e.g., cancellation).

Similarly, if the poset $\mathscr{A}$ is a generalized *commutative* difference algebra $(\mathscr{A}, \ominus, 0, \le)$ then

$$p(a, b, c) = \begin{cases} 1 & \text{iff } a \le c \ominus b \\ 0 & \text{else,} \end{cases}$$

and

$$j(a) = \begin{cases} 1 & \text{iff } 0 \le a \\ 0 & \text{else,} \end{cases}$$

yield an (associative and unital) promonoidal category $(\mathscr{A}, p, j)$. Note that a *commutative* generalized effect algebra is related to a (commutative) generalized difference algebra by

$$a \oplus b \le c \text{ iff } a \le c \ominus b.$$

The key feature regarding [5] (Riecanová) is that the promonoidal category

$$\mathscr{P} = \mathscr{A} + \mathscr{A}^{\mathrm{op}}$$

constructed in the embedding theorem [5, Proposition 21.2.4] (due to Hedliková and Pulmannová) is in fact $*$-autonomous under the switch map

$$S : (\mathscr{A} + \mathscr{A}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{A}^{\mathrm{op}} + \mathscr{A} \cong \mathscr{A} + \mathscr{A}^{\mathrm{op}}.$$

Again the convolution

$$\begin{aligned} [\mathscr{P}, \mathscr{V}] &= [\mathscr{A} + \mathscr{A}^{\mathrm{op}}, \mathscr{V}] \\ &= [\mathscr{A}, \mathscr{V}] \times [\mathscr{A}^{\mathrm{op}}, \mathscr{V}] \end{aligned}$$

is $*$-autonomous monoidal and complete.

Remark. The construction of this $\mathscr{P}$ is closely related to (but seems not the same as) that of the free $*$-autonomous promonoidal category on a given promonoidal category due to Luigi Santocanale (unpublished?), the latter giving the "Chu construction" upon convolution with $\mathscr{V}$.

Example 7 (Conformal field theory)**.** An example of a different nature ($\mathscr{V} = \mathbf{Vect}$) arises in RCFT [6, 4.17] as a **Vect** promonoidal structure on a (discrete) finite set $\mathscr{A}$ with a distinguished base point 0.

The promultiplication

$$p : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \longrightarrow \mathbf{Vect}_{\mathrm{fd}}$$

is given by

$$p(i, j, k) = V_{ij}^k = N_{ij}^k \cdot K, \quad \mathscr{A}(i, j) = \delta_{ij} \cdot K, \quad \text{and} \quad j(i) = \delta_{0i} \cdot K.$$

Symmetry is described by a set of (coherent) isomorphisms

$$p(i, j, k) \cong p(j, i, k),$$

associativity by isomorphisms

$$\bigoplus_x p(i, j, x) \otimes p(x, k, l) \cong \bigoplus_x p(i, x, l) \otimes p(j, k, x),$$

and $*$-autonomy by the cyclic condition

$$p(i, j, Sk) = p(j, k, Si) = p(k, i, Sj)$$

where

$$S : \mathscr{A} \longrightarrow \mathscr{A}, S^2 = 1$$

is the involution of the RFCT.

Here we insist also that

$$p(Si, Sj, Sk) = p(i, j, k)^*,$$

where $p(i, j, k)^*$ is the dual space of $p(i, j, k)$.

A useful way of abstracting this situation, especially for the purposes of constructing rigid convolutions of the form $[\mathscr{A}, \mathbf{Vect}_{\mathrm{fd}}]$, is to replace the finite set $\mathscr{A}$ above by any promonoidal $\mathbf{Vect}_{\mathrm{fd}}$-category $(\mathscr{A}, p, j)$ with ob$\mathscr{A}$ finite; then this $\mathscr{A}$ has a distinguished base object if $\mathscr{A}$ has an identity object representing j.

Remark.

(i) $*$-autonomous monoidal categories (under that name) were introduced by M. Barr, and then studied extensively. Their relationship to classical Frobenius structures was recognized in [3], and then made explicit in [8].

(ii) The $*$-autonomous structure on the extended real numbers was noted (by the author) in the context of a lecture entitled "$*$-autonomous convolution" (Australian Category Seminar, March 5, 1999), and recently introduced anew by M. Grandis.

(iii) The examples above admit generalizations to more elaborate promonoidal settings. Are there corresponding physical interpretations?

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